MATHEMATICS SPECIALIST

MAWA Semester 1 (Unit 3) Examination 2017

Calculator-assumed

Marking Key

Section Two: Calculator-assumed (92 Marks)

Question 8(a)

Solution $\boldsymbol{r} = \overrightarrow{OP} + t\overrightarrow{PQ} = 2\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k} + t(3\boldsymbol{i} + 4\boldsymbol{j} - 8\boldsymbol{k})$	
Marking key/mathematical behaviours	Marks
• uses $\boldsymbol{r} = \overrightarrow{OP} + t\overrightarrow{PQ}$ (or $\boldsymbol{r} = \overrightarrow{OQ} + t\overrightarrow{QP}$) • evaluates $\overrightarrow{PQ}$ correctly • writes vector eqn of line	1 1 1

Question 8(b)

Solution $x = 2 + 3t, y = -3 + 4t$ and $z = 5 - 8t$	
Marking key/mathematical behaviours	Marks
• chose correct format for parametric equations • correct parametric equations	1 1

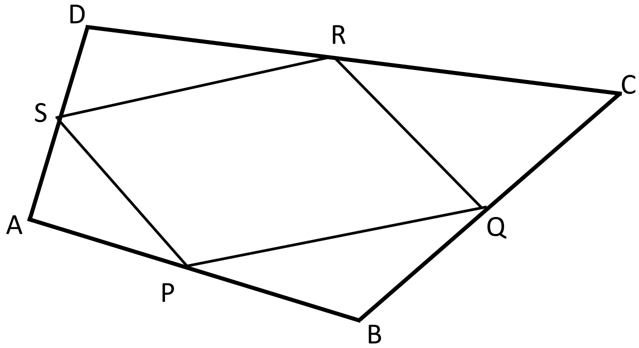
Question 8(c)

Solution R lies on $\ell \Longrightarrow 8 = 2 + 3t,\ 5 = -3 + 4t$ and $0 = 5 - 8t$ for some t From equation 1, $t = 2$. But $t = 2$ is not a solution of equation 3. So R does not lie on ℓ	
Marking key/mathematical behaviours	Marks
• obtains $8 = 2 + 3t,\ 5 = -3 + 4t$ and $0 = 5 - 8t$ • shows that there is no simultaneous solution for t • obtains the correct answer	1 1 1

Question 9(a)

Solution

$$\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ}$$
$$= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$$
$$= \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right)$$
$$= \frac{1}{2}\overrightarrow{AC}$$
$$\overrightarrow{SR} = \overrightarrow{SD} + \overrightarrow{DR}$$
$$= \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC}$$
$$= \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{DC}\right)$$
$$= \frac{1}{2}\overrightarrow{AC}$$
$$\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC} = \overrightarrow{SR}$$

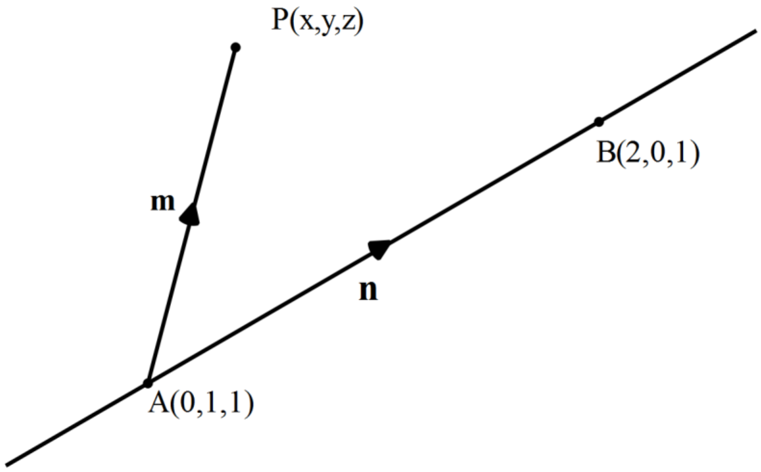


So the opposite sides PQ and SR in the quadrilateral PQRS are parallel (and equal in length)
By a similar argument, the sides QR and PS are parallel.
Since both pairs of opposite sides are parallel, PQRS is a quadrilateral

Marking key/mathematical behaviours	Marks
• uses $\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ}$ • uses mid-point property to establish $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$ • shows that PQ and SR are parallel • deduces that QR and PS are parallel • states that quad is a parallelogram	1 1 1 1 1

Question 9(b)

Solution



$$\mathbf{n}=\begin{pmatrix}2\\-1\\0\end{pmatrix} \quad \text{and} \quad \mathbf{m}=\begin{pmatrix}x\\y-1\\z-1\end{pmatrix}$$

$$\therefore \mathbf{n}\times\mathbf{m}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & -1 & 0\\x & y-1 & z-1\end{vmatrix}=\begin{pmatrix}-1\times(z-1)-0\\2(z-1)-0\\2(y-1)-(-1)(x)\end{pmatrix}=\begin{pmatrix}1-z\\2-2z\\x+2y-2\end{pmatrix}$$

Marking key/mathematical behaviours	Marks
• draws a diagram to illustrate the situation	1
• evaluates $\mathbf{n}$	1
• determines $\mathbf{m}$ in terms of x, y and z	1
• shows how each of the components of the cross product are calculated	1
• calculates each correctly	1

Question 10(a)

Solution

$a = 6 \quad \Rightarrow P(x) = (x^2 + 2x + b)(x - 6)$

$\Rightarrow P(x)$ has roots at $x = 6$ and at $x^2 + 2x + b = 0$

$x^2 + 2x + b = 0 \Rightarrow x = \dfrac{-2 \pm \sqrt{4 - 4b}}{2} \Leftrightarrow x = -1 \pm \sqrt{1 - b}$

So the possible roots of $P(x) = 0$ are 6, $-1 \pm \sqrt{1 - b}$

$a = 6, \quad \Rightarrow 0 \le b \le 6$

For,

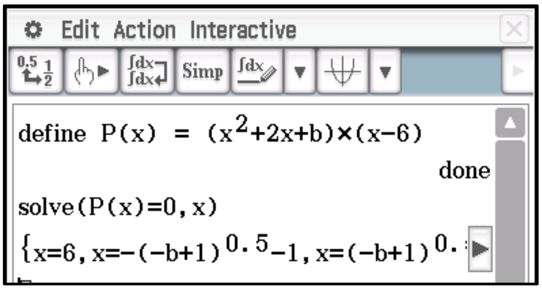
$b = 0,\ P(x) = x(x + 2)(x - 6) \Rightarrow$ roots are $-2, 0, 6$

$b = 1,\ P(x) = (x^2 + 2x + 1)(x - 6) = (x + 1)^2(x - 6) \Rightarrow$ roots are $-1, -1, 6$

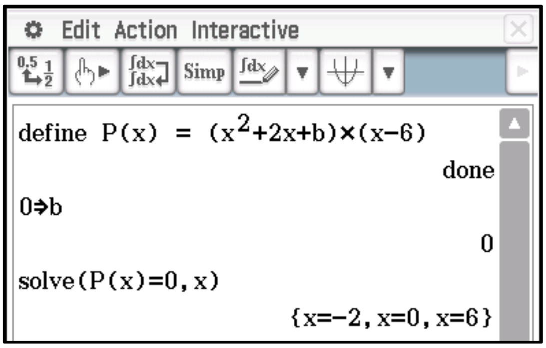
$b > 1,\ \sqrt{1 - b}$ is complex $\Rightarrow$ no other real roots

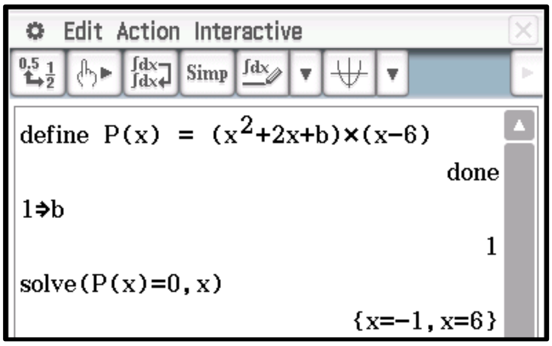
So all the possible real roots of $P(x) = 0$ are $-1, -1, 6$ for $b = 1$ or $-2, -0, 6$ for $b = 0$

Or using a CAS calculator



Substituting $b = 0$ or $b = 1$ into $P(x) = 0$ and solving for x





Marking key/mathematical behaviours	Marks
• Substitutes $a = 6$ into $P(x)$ and indicates $x = 6$ is a root	1
• Applies the quadratic formula to determine the roots of $x^2 + 2x + b = 0$ Or states them by use of a graphic calculator	1
• Indicates that $b = 0$ or 1 only, for real roots	1
• Determines the set of roots for $b = 0$	1
• Determines the set of roots for b=1	1

Question 10(b)

Solution

Given $x^2+2x+b=\left(x-\left(-1+\sqrt{1-b}\right)\right)\left(x-\left(-1-\sqrt{1-b}\right)\right)$

For $b=2$, $x^2+2x+b=\left(x-\left(-1+\sqrt{-1}\right)\right)\left(x-\left(-1-\sqrt{-1}\right)\right)=\left(x-(-1+i)\right)\left(x-(-1-i)\right)$

Here $\alpha=-1$ and $\beta=\pm1$ - so both rational

For $b=3$, $\sqrt{1-b}=\sqrt{-2}=\sqrt{2}i$ so β is not rational. Similarly, for $b=4$.

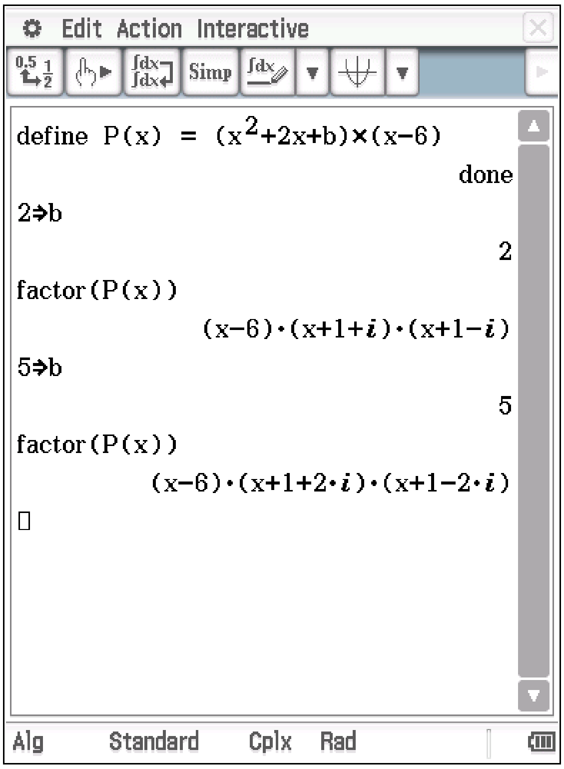
For $b=5$, $\sqrt{1-b}=\sqrt{-4}=2i$ and so $x^2+2x+b==\left(x-(-1+2i)\right)\left(x-(-1-2i)\right)$

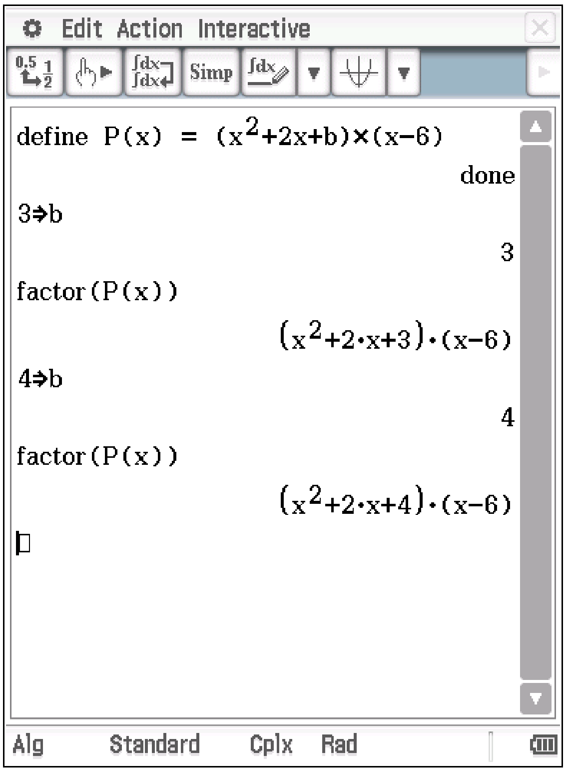
Here $\alpha=-1$ and $\beta=\pm2$ - so again, both rational

Hence, $P(x)=\left(x-(-1+i)\right)\left(x-(-1-i)\right)(x-6)$ for $b=2$

And $P(x)=\left(x-(-1+2i)\right)\left(x-(-1-2i)\right)(x-6)$ for $b=5$

Or using a CAS calculator





Marking key/mathematical behaviours	Marks
• Determines the complex linear factors of x^2+2x+b for $b=2$ or $b=5$ in the required form	1
• shows that α and β are rational (by finding them)	1
• Repeats procedure for the other value of b	1
• Shows that β is irrational for $b=3$ and for $b=4$	1
• States the three linear, rational factors of $P(x)$ for $b=2$ and $b=5$	1

Question 11(a)

Solution

Let $z = a + bi$

$|z - 2i| = |z - 2|$

$\Rightarrow \quad |a + bi - 2i| = |a + bi - 2|$

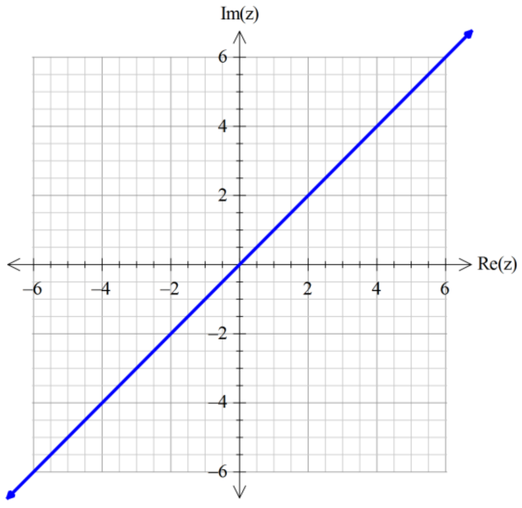
$\Rightarrow \quad |a - (2 - b)i| = |(a - 2) + bi|$

$\Rightarrow \quad \sqrt{a^2 + (2 - b)^2} = \sqrt{(a - 2)^2 + b^2}$

$\Rightarrow \not{a}^2 + \not{4} - 4b + \not{b}^2 = \not{a}^2 - 4a + \not{4} + \not{b}^2$

$\Rightarrow \quad b = a$

$\Rightarrow \mathrm{Im}(z) = \mathrm{Re}(z)$

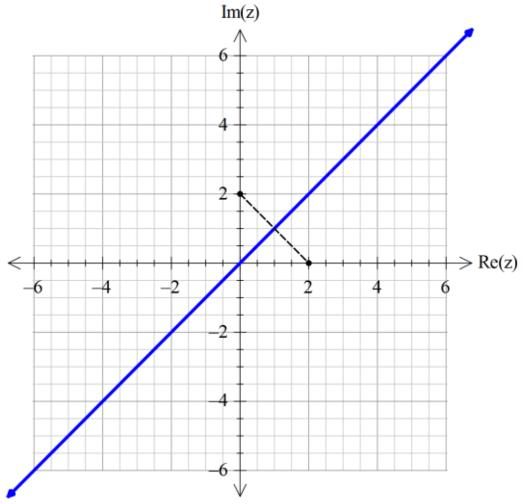


Alternatively,

Note that the required solution set is the perpendicular bisector of the line segment joining the points $(0, 2)$ and $(2, 0)$

Plot $(0, 2)$ and $(2, 0)$

Draw the perpendicular bisector



Marking key/mathematical behaviours	Marks
• Calculates the magnitude of each part and equates	1
• Determines $\mathrm{Im}(z) = \mathrm{Re}(z)$	1
• Graphs the line $\mathrm{Im}(z) = \mathrm{Re}(z)$	1
Alternatively,	
• Plots the points $(0, 2)$ and $(2, 0)$	1
• Indicates that the perpendicular bisector of the line segment joining these points is the solutions set required	1
• Draws the bisector accurately	1

Question 11(b)

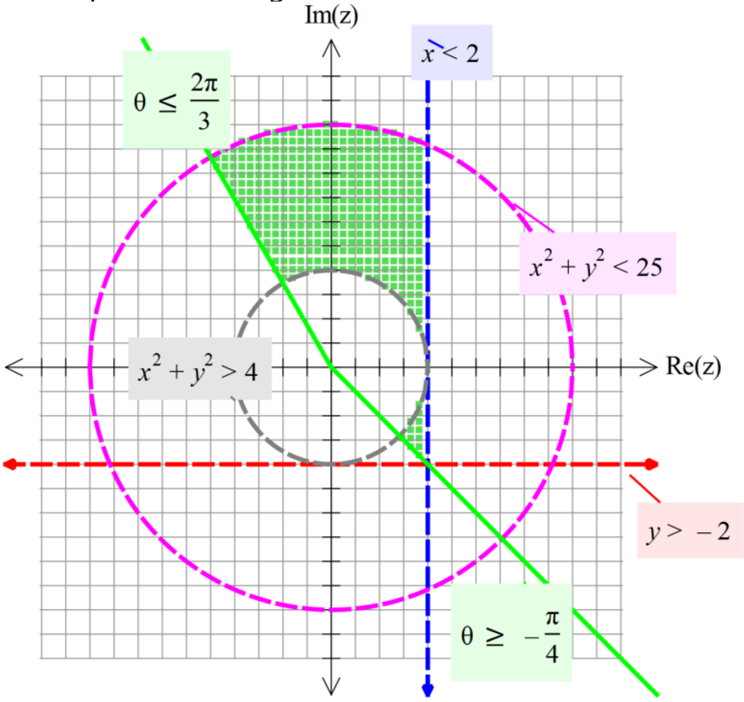
Solution

$\text{Re}(z) < 1,\ \ \text{Im}(z) > -2,\ \ 1 < |z| < 5 \text{ and } -\dfrac{\pi}{4} \le Arg(z) \le \dfrac{2\pi}{3}$

Draw each of the given inequalities

indicate which are dotted lines and which include the points of the line or circle.

Shade the regions that meet all the conditions.



Marking key/mathematical behaviours	Marks
• Each inequality graphed correctly (without the shading)	6
• Correct regions that meets the conditions	1
• Correct boundaries for these regions	1

Question 12

Solution

$z^5 = -32i$

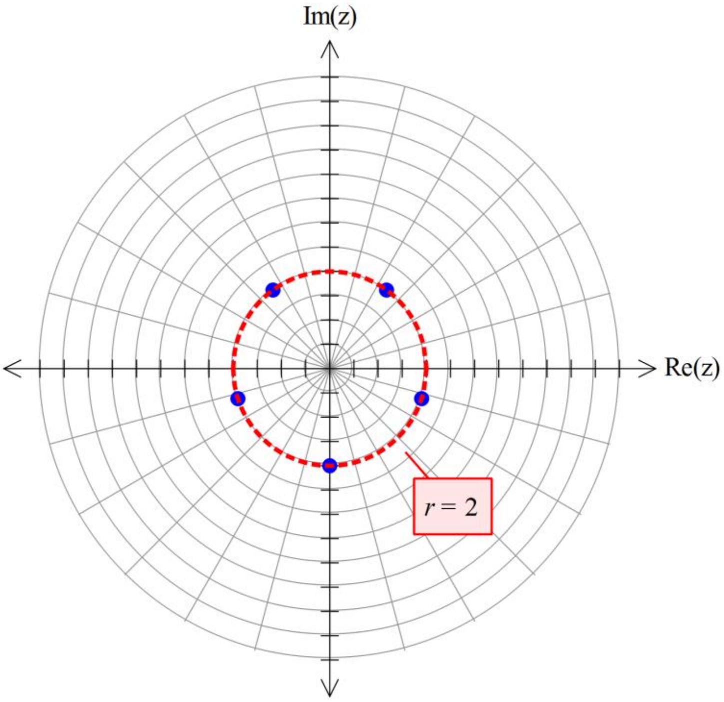
$\Rightarrow -32i = 32cis\left(-\frac{\pi}{2}\right)$

$\Rightarrow r^5 = 32,$ so $r = 2$, and

$\theta = \frac{-\frac{\pi}{2} + 2\pi k}{5} \Rightarrow \theta = -\frac{9\pi}{10}, -\frac{5\pi}{10}, -\frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}$

So the 5th roots of $-32i$ are:

$2cis\left(-\frac{9\pi}{10}\right),\ 2cis\left(-\frac{5\pi}{10}\right),\ 2cis\left(-\frac{\pi}{10}\right),\ 2cis\left(\frac{3\pi}{10}\right),\ 2cis\left(\frac{7\pi}{10}\right),$



Marking key/mathematical behaviours	Marks
• converts $-32i$ in polar form correctly	1
• Determines $r = 2$ and $\theta = -\frac{\pi}{10}$	1
• Determines the other four values of θ	1
• Represents the five values of θ on the Argand plane	1
• Accurately places the roots on a circle with a scale indicating the radius	1

Question 13

Solution	
Let $z = a + bi,\ \ \bar{z} = a - bi$ $2\bar{z} + z\bar{z} = 24 + 8i$ $\Rightarrow 2(a - bi) - (a + bi)(a - bi) = 24 + 8i$ So $\Rightarrow 2a + (a^2 + b^2) = 24$ and $-2b = 8$ (comparing real and imaginary parts) $\Rightarrow b = -4$ and $a^2 + 2a - 8 = 0$ $\Rightarrow b = -4$ and $a = -4, 2$ Hence, $z = -4 - 4i$ or $2 - 4i$	
Marking key/mathematical behaviours	Marks
• Substitutes $z = a + bi$ into given equation • Substitutes $\bar{z} = a - bi$ into given equation • Compares real and imaginary parts • Determines $b = -4$ • Determines the two values for a • States values of z	1 1 1 1 1 1

Question 14(a)

Solution	
$\overrightarrow{AB} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{AC} = 3\boldsymbol{i} + \boldsymbol{k}$ lie in $\mathcal{P}$ so $\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (3\boldsymbol{i} + 4\boldsymbol{j}) \times (3\boldsymbol{i} + \boldsymbol{k}) = 4\boldsymbol{i} - 3\boldsymbol{j} - 12\boldsymbol{k}$ is normal to $\mathcal{P}$	
Marking key/mathematical behaviours	Marks
• obtains 2 non-parallel vectors in $\mathcal{P}$ • calculates the cross product correctly	1 1

Question 14(b)

Solution	
Vector equation $\boldsymbol{r} \cdot \boldsymbol{n} = c$ for $\mathcal{P}$ is $4x - 3y - 12z = 4 \times -3 = -12$	
Marking key/mathematical behaviours	Marks
• obtains a vector equation • obtains a Cartesian equation	1 1

Question 14(c)

Solution At the point of intersection $4(3-3t)-3(5+2t)-12t=-12$ i.e. $-3-30t=-12$, i.e. $t=0.3$ So the point of intersection has coordinates $(3-0.9,5+0.6,0.3)=(2.1,5.6,0.3)$	
Marking key/mathematical behaviours	Marks
• substitutes for $\boldsymbol{r}$ in vector equation of $\mathcal{P}$ • solves for t correctly • states correct coordinates	1 1 1

Question 15(a)

Solution $x=3\cos kt$ and $y=-2\sin kt$, so $\left(\frac{x}{3}\right)^2+\left(\frac{y}{-2}\right)^{\mathbf{2}}=\cos^2 kt+\sin^2 kt=1$ i.e. $\frac{x^2}{9}+\frac{y^2}{4}=1$	
Marking key/mathematical behaviours	Marks
• obtains the formulae for x and y • uses trig identity • eliminates t correctly	1 1 1

Question 15(b)

Solution The path is an ellipse.	
Marking key/mathematical behaviours	Marks
• obtains correct answer	1

Question 15(c)

Solution $\boldsymbol{v}(t)=-3k\sin kt\,\boldsymbol{i}-2k\cos kt\;\boldsymbol{j}$, and $\boldsymbol{a}(t)=-3k^2\cos kt\,\boldsymbol{i}+2k^2\sin kt\;\boldsymbol{j}$,	
Marking key/mathematical behaviours	Marks
• correct answer for $\boldsymbol{v}(t)$ • correct answer for $\boldsymbol{a}(t)$	1 1

Question 15(d)

Solution $\boldsymbol{r}(0) = 3\,\boldsymbol{i}$ and $v(0) = -2k\,\boldsymbol{j}$ So at (3,0), the extreme 'easterly' end of the path, the train is moving 'south', i.e. in the 'negative' direction of the y axis. So the train is moving in a clockwise direction around the closed path.	
Marking key/mathematical behaviours	Marks
• obtains the correct answer • gives a valid reason	1 1

Question 15(e)

Solution $\boldsymbol{v}(t)\cdot\boldsymbol{a}(t) = 9k^3\sin kt\cos kt - 4k^3\sin kt\cos kt = 5k^3\sin kt\cos kt$ So $\boldsymbol{v}(t)\cdot\boldsymbol{a}(t) = 0$ when $\sin kt = 0$ or $\cos kt = 0$ $\sin kt = 0 \Leftrightarrow (x, y) = (\pm 3, 0)$ and $\cos kt = 0 \Leftrightarrow (x, y) = (0, \pm 2)$ So the velocity and acceleration are mutually perpendicular at the points $(3,0), (-3,0), (0,2)$ and $(0,-2)$.	
Marking key/mathematical behaviours	Marks
• uses $\boldsymbol{v}(t)\cdot\boldsymbol{a}(t) = 0 \Leftrightarrow$ perpendicularity • obtains $\boldsymbol{v}(t)\cdot\boldsymbol{a}(t)$ correctly • obtains correct answers	1 1 1

Question 15(f)

Solution $92k = 2\pi$ so $k = 0.0683$, correct to 3 significant figures	
Marking key/mathematical behaviours	Marks
• uses $92k = 2\pi$ • obtains solution, correct to 3 sig. figs.	1 1

Question 15(g)

Solution $\boldsymbol{v}(t) = -0.205\sin 0.0683t\,\boldsymbol{i} - 0.137\cos 0.0683t\,\boldsymbol{j}$, (from (c) and (f)) so $v(t)^2 = (-0.205)^2\sin^2 0.0683t + (-0.137)^2\cos^2 0.0683t$ $= (-0.205)^2 - 0.0233\cos^2 0.0683t$ So $v_{max} = 0.205$ i.e. the maximum speed is 20.5 centimetres per second	
Marking key/mathematical behaviours	Marks
• obtains correct expression for $\boldsymbol{v}(t)$ • recognizes speed as the length of the velocity vector • obtains the correct answer	1 1 1

Question 16(a)

Solution $x^2 - 6x + y^2 + z^2 + 10z = 2 \Leftrightarrow (x-3)^2 + y^2 + (z+5)^2 = 2 + 9 + 25 = 36 = 6^2$ So the radius is 6 and the centre C has coordinates $(3,0,-5)$	
Marking key/mathematical behaviours	Marks
• completes the square • obtains correct radius • obtains correct coordinates of C	1 1 1

Question 16(b)

Solution Substituting $(x,y,z) = (9+2t, -2t, 1+t)$ in the equation of the sphere gives $(9+2t)^2 - 6(9+2t) + (-2t)^2 + (1+t)^2 + 10(1+t) = 2$, i.e. $38 + 36t + 9t^2 = 2$, i.e. $9(t+2)^2 = 0$, i.e. $t = -2$ (or by calculator) so ℓ and $\mathcal{S}$ intersect so $(x,y,z) = (5,4,-1)$ at the only point of intersection	
Marking key/mathematical behaviours	Marks
• substitutes correctly • sets up an equation for t • solves for t • deduces that the line and sphere intersect • solves for the coordinates x, y and z at the only point of intersection	1 1 1 1 1

Question 16(c)

Solution If P is the point of intersection $CP = 2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$, The vector $2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ is parallel to the line ℓ, And $(2\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}) \cdot (2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = 4 - 8 + 4 = 0$ So the direction of the line is perpendicular to the radial vector CP. Since P is the point of intersection of ℓ and $\mathcal{S}$, ℓ must be tangential to $\mathcal{S}$ at P.	
Marking key/mathematical behaviours	Marks
• shows that ℓ is perpendicular to the radial vector at the point of intersection • argues that this implies the tangency property for ℓ	1 1

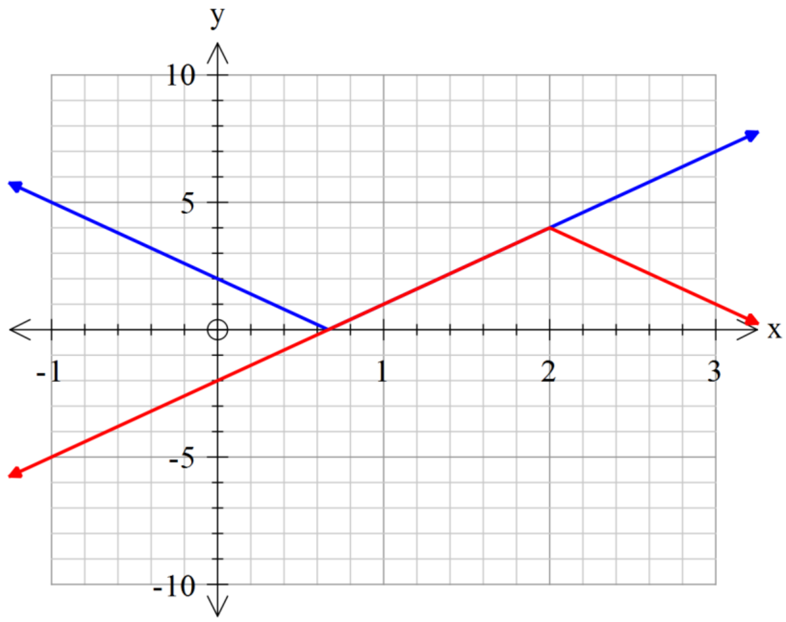
Question 17(a)

Solution	
The midpoint of -8 and 2 on the real number line is -3. $x \le -3$	
Marking key/mathematical behaviours	Marks
• uses points -8,2 • obtains correct answer	1 1

Question 17(b)

Solution	
$\lvert 2x-a\rvert \le \lvert x-7\rvert$ At $x=-5$, $\lvert x-7\rvert = 12$ $\lvert 2x-a\rvert = \lvert -10-a\rvert = 12 \quad \Rightarrow \quad a=2$ or $a=-22$ At $x=3$, $\lvert x-7\rvert = 4$ $\lvert 2x-a\rvert = \lvert 6-a\rvert = 4 \quad \Rightarrow \quad a=2$ or $a=10$ Hence, $a=2$	
Marking key/mathematical behaviours	Marks
• uses endpoints of inequality • solves for a • chooses correct value for a	1 1 1

Question 17(c)

Solution	
Graphs overlap on the interval $\frac{2}{3} \le x \le 2$. The graph of $y = 3\|x\|$ has been translated horizontally 2 units in the positive x direction, reflected in the horizontal axis and then translated vertically 4 units in the positive y direction. Hence $p = -3, q = -2$ and $s = 4$. 	
Marking key/mathematical behaviours	**Marks**
• recognizes that graphs overlap on given interval only • obtains -3 for p • obtains q=-2 • obtains s=4	1 1 1 1